Block-Structure Method for the Solution of the Matrix System of Equations $g_{ij}g^{jk} = \delta_i^k$ in the N−dimensional Case

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Abstract

In this paper a new block-structure method is presented for the solution of the well-known from gravity theory matrix system of equations $g_{ij}g^{jk} = \delta_i^k$ (with respect to the unknown covariant components g_{ij} and by known contravariant ones g^{jk}) by transforming this matrix system into a linear algebraic system of equations in the general N−dimensional case. Although powerful computer methods exist for the solution of this problem for a given (fixed) dimension of the matrices g^{ij} and especially for numerical elements of g^{ij}, the structure of the obtained linear algebraic system in the general N−dimensional case and for arbitrary elements of g^{ij} (functions) has not been known.

The proposed new analytical block-structure method for the case of symmetrical matrices g_{ij} and g^{jk} (the standard case in gravity theory) is based on the construction of a block-structure matrix, whose "elements" are again matrices. The method allows to obtain the structure of this linear system in the general N−dimensional case, after multiplication (to the left) with the transponed matrix.

Some arguments are given why the proposed method may be applied, after some refinement and generalization for the case of non-symmetrical matrices g_{ij} and g^{jk}, for finding the graviton modes in the Kaluza-Klein expansion in theories with extra dimensions.

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1 INTRODUCTION

The system of equations $g_{ij}g^{jk} = \delta_i^k$ is well-known and important in gravity theory, since from it the contravariant metric tensor components are determined under known covariant ones. In spite of its importance, this system is not well-understood as a mathematical object. For example, in Bergman's book on General Relativity [1] (Ch. 5, eq. 5.64) it is written that *" if the determinant g_{ij} is not equal to zero, then a multitude of new variables g^{jk} can be introduced according to the relation $g_{ij}\widetilde{g}^{jk} = \delta_i^k$ "*. No doubt, similar statements can be seen in other books. Bergmann's statement would have been true if this system is a trivial linear algebraic system of equations with a number of equations equal to the number of variables. But in fact, this is a *predetermined system of equations* with a number of equations n^2 greater than the number $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$ of unknown variables. This is the first reason why Bergmann' s statement does not hold. Secondly, *this is not a linear algebraic system of equations, but an operator (matrix) system of equations* [2] of the type $AX = B$, where X is no longer a vector-column, but a matrix. Consequently, the linear algebra theorems do not hold any more.

One of the basic new results in the present paper is that the initially given operator system of equations can be reduced to a linear algebraic system of equations $B\widetilde{Y} = T$, where the unknown elements $\widetilde{Y}$ represent the matrix elements X, arranged in the form of a vector-column. At first, the system will be written in the form $\widetilde{A}\widetilde{Y} = \delta_i^k$, where $\widetilde{A}$ is a non-quadratic $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n \right] \times n^2$ matrix. Further, non-quadratic matrices will be denoted as $a \times b$, where a will be the number of rows in the matrix and b will be the number of columns.

In the conclusion, a possible application of the proposed mathematical method will be briefly commented in reference to the problem about finding the graviton's modes in the Kaluza-Klein's expansion under toroidal compactification in theories with extra dimensions.

2 Some general properties of the system $g_{ij}\widetilde{g}^{jk} = \delta_i^k$ in the general $n-$dimensional case.

Since the number of equations is greater than the number of variables, from the n^2 equations one can select $n^2 - n$ equations with different values of i and k, for which the right-hand side (R. H. S.) will be zero. Yet the number of the chosen equations remains to be greater than the number of variables $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$, which is confirmed by the equality

$$(n^2 - n) - \left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n \right] = \frac{n(n-3)}{2} \quad , \tag{2.1}$$

fulfilled for $n > 3$. Only for the case $n = 3$, the number of the equations $n^2 - n = 6$ with a zero R. H. S. becomes exactly equal to the number of variables $\begin{pmatrix} n \\ 2 \end{pmatrix} + n = \frac{n(n+1)}{2} = \frac{3 \cdot 4}{2} = 6$. Therefore for the case of an arbitrary n from these $n^2 - n$ equations one can choose $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n \right]$ equations. In each row of the $\left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n \right] \times \left[\begin{pmatrix} n \\ 2 \end{pmatrix} + n \right]$ matrix of coefficient functions of this system there will be only n functions $\widetilde{g}^{jk}$, the rest $\begin{pmatrix} n \\ 2 \end{pmatrix}$ of the elements will be zero. The determinant of the matrix will be equal to zero, a proof of which in the general case of an arbitrary n (and also for $n = 3$) will be presented in the next sections. Therefore, the solutions g_{ij} of this $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$ dimensional homogeneous system of equations with a zero determinant are arbitrary.

Now we are left with $\frac{n(n-3)}{2}$ equations (again with $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$ variables) with a zero R. H. S. (we shall call it the first system of equations) and with another n equations (the second system) with a R. H. S. of each equation, equal to 1. Since the number of variables $\begin{pmatrix} n \\ 2 \end{pmatrix} + n$ in the first system is greater than the number $\frac{n(n-3)}{2}$ of equations and

$$\begin{pmatrix} n \\ 2 \end{pmatrix} + n - \frac{n(n-3)}{2} = 2n \quad , \tag{2.2}$$

again one can treat as unknown only $\frac{n(n-3)}{2}$ variables and transfer the rest $2n$ variables in the R. H. S., which will become different from zero. Analogously, for the second system one can treat as unknown only n variables and transfer the rest $\begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n(n-1)}{2}$ variables in the R. H. S. If this R. H. S. is different from zero and moreover, the determinant of the coefficient functions $\widetilde{g}^{ij}$ is also different from zero, then one can find unique solutions for these n variables g_{ij}. Now comes the most important point of the proof: Since

$$n + \frac{n(n-1)}{2} > 2n \quad , \tag{2.3}$$

one can take all n of the uniquely found solutions plus n more unfixed (freely varied) variables from the R. H.S. of the second system and "plunge" them into the R. H. S. of the first system. Let us remember that the first system has $\frac{n(n-3)}{2}$ unknown variables, but one may note that $\begin{pmatrix} n \\ 2 \end{pmatrix} + n - 2n = \frac{n(n-3)}{2}$ variables from the second system have not been transferred in the R. H. S. of the first system. Therefore, one can choose these $\frac{n(n-3)}{2}$ variables to be the unknown variables for the first system, and if the determinant of the coefficient functions is nonzero, an unique solution can be found for them.

As a whole , one would have a maximum of $n + \frac{n(n-3)}{2} = \frac{n(n-1)}{2}$ uniquely fixed variables and the rest of the variables may be varied freely.

3   BLOCK STRUCTURE METHOD FOR   THE PARTIAL $n = 3$ CASE

The purpose of the present section will be to develop a method for solving the system of equations $g_{ij}g^{jk} = \delta_i^k$ for the case $n = 3$. Since in principle the system for $n = 3$ can be solved in an elementary manner, the aim will be not to find another more convenient method, but rather than that find a method, which can further be generalized to higher dimensions.

In matrix notations for the unknown variables $g_{11}, g_{22}, g_{33}, g_{12}, g_{23}$ and g_{13} the system $g_{ij}g^{jk} = \delta_i^k$ of nine equations (for the different indices i and k) can be written as

$$\widetilde{A}X = E \quad , \tag{3.1}$$

where X^T further will denote the transponed vector

$$X^T \equiv (g_{11}, g_{22}, g_{33}, g_{12}, g_{23}, g_{13}) \quad . \tag{3.2}$$

and E^T - the transponed 9− vector:

$$E^T = (1, 0, 0, 0, 1, 1, 0, 0, 1) \quad . \tag{3.3}$$

The 6×9 matrix $\widetilde{A}$ has the following interesting block structure:

$$\widetilde{A} \equiv \begin{pmatrix} P_1 & Q_1 \\ P_2 & Q_2 \\ P_3 & Q_3 \end{pmatrix} \quad , \tag{3.4}$$

where $(s = 1, 2, 3)$ the matrices P_s and Q_s are the following:

$$P_s \equiv \begin{pmatrix} g^{s1} & g^{s2} & g^{s3} \\ 0 & g^{s1} & 0 \\ 0 & 0 & g^{s1} \end{pmatrix} \quad ; \quad Q_s \equiv \begin{pmatrix} 0 & 0 & 0 \\ g^{2s} & g^{3s} & 0 \\ 0 & g^{2s} & g^{3s} \end{pmatrix} \quad . \tag{3.5}$$

In order to find the solution $X = \widetilde{A}^{-1}E$ of the system (3.1), one has to find the inverse matrix $\widetilde{A}^{-1}$. For the case of quadratic matrices with the block structure

$$M \equiv \begin{pmatrix} A & B \\ C & D \end{pmatrix} \quad , \tag{3.6}$$

where A, B, C and D are $n \times n, q \times n, n \times q$ and $q \times q$ matrices correspondingly, the so called Frobenius formulae [2] for finding the inverse matrix M^{-1} is valid

$$M^{-1} = \begin{pmatrix} A^{-1} + A^{-1}BH^{-1}CA^{-1} & -A^{-1}BH^{-1} \\ -H^{-1}CA^{-1} & H^{-1} \end{pmatrix} \quad , \tag{3.7}$$

where H is the matrix:

$$H \equiv D - CA^{-1}B \ . \tag{3.8}$$

In the present case, the Frobenius formulae cannot be applied to the block matrix (3.4), since it is not a quadratic one. However, if X is a solution of the system (3.1), then it is a solution also of the equation $\widetilde{A}^T \widetilde{A} X = E \widetilde{A}^T$, where the 6×9 matrix $\widetilde{A}$, multiplied to the left with its transponed one, gives already the quadratic 9×9 matrix $\widetilde{A}^T \widetilde{A}$. Further it shall be demonstrated how the Frobenius inversion formulae can be applied twice in respect to $\widetilde{A}^T \widetilde{A}$.

The matrix $\widetilde{A}^T \widetilde{A}$ can be calculated to be the following block matrix:

$$\widetilde{A}^T \widetilde{A} = \begin{pmatrix} \sum\limits_{i=1}^{3} P_i^T P_i & \sum\limits_{j=1}^{3} P_j^T Q_j \\ \sum\limits_{k=1}^{3} Q_k^T P_k & \sum\limits_{l=1}^{3} Q_l^T Q_l \end{pmatrix} \tag{3.9}$$

and P_i, Q_j are the corresponding matrices (3.5) and their transponed ones. The block matrices in (3.9) are found to be the following 3×3 matrices, which shall further be identified with the corresponding block - matrices A, B, C and D in (3.6):

$$A \equiv \sum_{i=1}^{3} P_i^T P_i = \begin{pmatrix} \sum\limits_{i=1}^{3} (g^{i1})^2 & \sum\limits_{i=1}^{3} g^{i1} g^{i2} & \sum\limits_{i=1}^{3} g^{i1} g^{i3} \\ \sum\limits_{i=1}^{3} g^{i1} g^{i2} & \sum\limits_{i=1}^{3} [(g^{i1})^2 + (g^{i2})^2] & \sum\limits_{i=1}^{3} g^{i2} g^{i3} \\ \sum\limits_{i=1}^{3} g^{i1} g^{i3} & \sum\limits_{i=1}^{3} g^{i2} g^{i3} & \sum\limits_{i=1}^{3} [(g^{i1})^2 + (g^{i3})^2] \end{pmatrix} \quad , \tag{3.10}$$

$$B \equiv \sum_{j=1}^{3} P_j^T Q_j = \sum_{j=1}^{3} g^{j1} \begin{pmatrix} 0 & 0 & 0 \\ g^{2j} & g^{3j} & 0 \\ 0 & g^{j2} & g^{j3} \end{pmatrix} \quad , \tag{3.11}$$

$$C \equiv \sum_{k=1}^{3} Q_k^T P_k = \sum_{k=1}^{3} g^{k1} \begin{pmatrix} 0 & g^{k2} & 0 \\ 0 & g^{k3} & g^{k2} \\ 0 & 0 & g^{k3} \end{pmatrix} \quad , \tag{3.12}$$

$$D \equiv \sum_{l=1}^{3} Q_l^T Q_l = \begin{pmatrix} \sum\limits_{l=1}^{3} (g^{2l})^2 & \sum\limits_{l=1}^{3} g^{2l} g^{3l} & 0 \\ \sum\limits_{l=1}^{3} g^{2l} g^{3l} & \sum\limits_{l=1}^{3} [(g^{21})^2 + (g^{3l})^2] & \sum\limits_{l=1}^{3} g^{2l} g^{3l} \\ 0 & \sum\limits_{l=1}^{3} g^{2l} g^{3l} & \sum\limits_{i=1}^{3} (g^{3l})^2 \end{pmatrix} \quad . \tag{3.13}$$

Note that the diagonal block - matrices A and D have non - zero determinants (even if $g^{ij} = dX^i dX^j$), while the non - diagonal block - matrices B and D have zero - determinants. However, in order to apply the Frobenius formulae for inverting the matrix (3.9) it is sufficient to have as invertible only the matrix A (and of course the matrix H).

4 MODIFICATION OF THE BLOCK STRUCTURE OF THE MATRIX A- THE $n = 3$ CASE AND THE GENERAL $N-$DIMENSIONAL CASE

The above presented method has nevertheless the following shortcomings:

1. It deals with an rectangular $p \times q$ matrix A for a system of equations with $p = \begin{pmatrix} n \\ 2 \end{pmatrix} + n$ variables and $q = n^2$ equations. At the same time it would have been much better to deal with a quadratic matrix at the beginning.

2. The block - matrix A contains two types of matrices P_s and Q_s, while it would be more convenient to have just one type of an elementary "constituent" $E_k^{(i)}$ with a definite structure, where the indice i denotes the corresponding column in the block matrix A (i.e. the number of the column, containing block - matrices) and the indice k - the corresponding row of block - matrices.

3. The matrix Q_s, given by the formulae (3.4) has a zero determinant.

To avoid these shortcomings, let us define an extended $9-$ dimensional vector Y, whose transponed one is

$$Y^T \equiv (g_{11}, g_{12}, g_{13}, g_{21}, g_{22}, g_{23}, g_{31}, g_{32}, g_{33}) \quad , \tag{4.1}$$

where the elements g_{21}, g_{32} and g_{31} formally shall be considered unknown, although they are equal to their symmetric counterparts g_{12}, g_{23} and g_{13}. Then the system of equations can be written as

$$MY = \overline{1} \quad , \tag{4.2}$$

where $\overline{1}^T$ is the transponed 9 -dimensional vector

$$\overline{1}^T \equiv (1, 0, 0, 0, 1, 0, 0, 0, 1) \tag{4.3}$$

and the 9×9 matrix M has the following block structure

$$M \equiv \begin{pmatrix} E_1^{(1)} & E_1^{(2)} & E_1^{(3)} \\ E_2^{(1)} & E_2^{(2)} & E_2^{(3)} \\ E_3^{(1)} & E_3^{(2)} & E_3^{(3)} \end{pmatrix} \quad . \tag{4.4}$$

The elementary 3×3 block matrices $E_k^{(1)}$, $E_k^{(2)}$ and $E_k^{(3)}$ ($k = 1, 2, 3$ denotes the number of the row) in each column are the following

$$E_k^{(1)} \equiv \frac{1}{2} \begin{pmatrix} 2g^{k1} & g^{k2} & g^{k3} \\ 0 & g^{k1} & 0 \\ 0 & 0 & g^{k1} \end{pmatrix} \quad , \tag{4.5}$$

$$E_k^{(2)} \equiv \frac{1}{2} \begin{pmatrix} g^{2k} & 0 & 0 \\ g^{1k} & 2g^{2k} & g^{3k} \\ 0 & 0 & g^{2k} \end{pmatrix} \quad , \tag{4.6}$$

$$E_k^{(3)} \equiv \frac{1}{2} \begin{pmatrix} g^{3k} & 0 & 0 \\ 0 & g^{3k} & 0 \\ g^{1k} & g^{2k} & g^{3k} \end{pmatrix} \qquad . \tag{4.7}$$

From (4.5 - 4.7) it is seen that depending on the indice k of the elementary matrices row and of the indice (s) for the elementary matrices column, the matrix $E_k^{(s)}$ for the $n-$dimensional case can be written as

$$E_k^{(s)} \equiv \frac{1}{(n-1)} \begin{pmatrix} g^{ks} & 0... & .... & ... & 0 & 0 \\ 0 & g^{ks} & 0.. & ... & 0 & 0 \\ .... & .... & ... & .... & .... & ... \\ g^{k1} & g^{k2}. & .. & (n-1)g^{ks}.. & ... & g^{kn} \\ .... & ... & ... & .... & .... & .... \\ 0 & 0 & 0 & .... & 0 & g^{ks} \end{pmatrix} \qquad . \tag{4.8}$$

In other words, on the main diagonal of the matrix $E_k^{(s)}$ the elements g^{ks} are situated, on the $s-$th row - the elements $(\ g^{k1},\ \ g^{k2},.\ \ .. \ \ (n-1)g^{ks}..\ \ ... \ \ ,g^{kn}\ )$ with an element $(n-1)g^{ks}$ on the $s-$th row and $s-$th column. This block structure (with some slight modifications) shall be obtained also for the $n-$ dimensional case, and thus the $3-$ dimensional case really helps to make the corresponding generalization for the $n-$ dimensional case.

Another advantage of the block - matrix representation is that it gives a possibility to apply twice the Frobenius formulae. Correspondingly, in the $n-$ dimensional case the Frobenius formulae will be applied $(n-1)$ times.

5   BLOCK MATRIX STRUCTURE IN THE N-DIMENSIONAL CASE

Following the same algorithm as in the preceeding subsections, we shall try to find the block structure of the system of equations $g_{ij}g^{jk} = \delta_i^k$ in the $n-$dimensional case. For $i = k$ the system can be written as $AX_k = \begin{pmatrix} 0 \\ .. \\ 1 \\ ... \end{pmatrix}$, where "1" is on the $k-$th place, the vector X_k^T is

$$X_k^T \equiv (g_{k1}, g_{k2}, ..., g_{kn}) \tag{5.1}$$

and the matrix A is

$$A \equiv \begin{pmatrix} g^{11} & g^{12} & .... & g^{1n} \\ g^{21} & g^{22} & .... & g^{2n} \\ ..... & ..... & .... & ..... \\ g^{n1} & g^{n2} & .... & g^{nn} \end{pmatrix} \tag{5.2}$$

The corresponding vectors $X_1, X_2, ..., X_k$ represent the rows of the symmetric matrix N

$$N \equiv \begin{pmatrix} g_{11} & g_{12} & ...... & g_{1n} \\ g_{21} & g_{22} & ..... & g_{2n} \\ .... & .... & .... & ... \\ g_{n1} & g_{n2} & .... & g_{nn} \end{pmatrix} \quad , \tag{5.3}$$

in which the unknown variables are in the lower triangular (half) part of the matrix (denoted by N^{tr}).

Let us construct a $\frac{n(n+1)}{2} \times n^2$ dimensional matrix B, which will multiply a $\frac{n(n+1)}{2}$ dimensional vector Y, formed by joining all the consequent rows of the triangular matrix N^{tr}

$$Y \equiv (g_{11}, g_{12}, ..., g_{1n}, g_{22}, g_{23}, ..., g_{2n}, ..., g_{n1}, g_{n2}, ..., g_{nn}) \quad . \tag{5.4}$$

Correspondingly the matrix B will have the following block triangular structure:

$$B \equiv \begin{pmatrix} B_{11} & 0 & ...... & 0 \\ B_{21} & B_{22} & ..... & 0 \\ .... & .... & .... & ... \\ B_{n1} & B_{n2} & .... & B_{nn} \end{pmatrix} \quad , \tag{5.5}$$

where each of the block matrices B_{kk} on the diagonal is an $n \times (n-k+1)$ (i.e. $(n-k+1)$ columns and n rows) dimensional matrix, obtained from the matrix A by removing the first $(k-1)$ columns. For example, $B_{11} \equiv A$, but

$$B_{22} \equiv \begin{pmatrix} g^{12} & g^{13} & ...... & g^{1n} \\ g^{22} & g^{23} & ..... & g^{2n} \\ .... & .... & .... & ... \\ g^{n2} & g^{n3} & .... & g^{nn} \end{pmatrix} \quad . \tag{5.6}$$

The corresponding matrix B_{kk} will be

$$B_{kk} \equiv \begin{pmatrix} g^{1k} & g^{1,(k+1)} & .... & g^{1,n} \\ g^{2k} & g^{2,(k+1)} & .... & g^{2,n} \\ ... & .... & .... & ..... \\ g^{nk} & g^{n,(k+1)} & ... & g^{nn} \end{pmatrix} \quad . \tag{5.7}$$

The block - matrices B_{sk} $(s > k)$ are $n \times (n-k+1)$ dimensional ones with just one nonzero column (the $s-k+1$ column) with the elements $g^{k1}, g^{k2}, ...$

$$B_{sk} \equiv \begin{pmatrix} 0 & ... & g^{k1} & ... & 0 \\ 0 & .... & g^{k2} & .... & 0 \\ ... & .... & ... & ... & ... \\ 0 & ... & g^{k,(n-1)} & .... & 0 \\ 0 & ... & g^{k,n} & .... & 0 \end{pmatrix} \quad . \tag{5.8}$$

8

Let us now have a look at the complex of neighbouring block matrices around the main block diagonal

$$K^{(k)} \equiv \begin{pmatrix} B_{k-1,k-1} & B_{k-1,k} \\ B_{k,k-1} & B_{kk} \end{pmatrix} \quad . \tag{5.9}$$

For illustration of the block matrix multiplication and in order to derive some useful formulaes, let us calculate $K^{(p)T} K^{(k)}$, which will be equal to

$$\begin{pmatrix} B_{p-1,p-1}^T B_{k-1,k-1} + B_{p,p-1}^T B_{k,k-1} & B_{p,p-1}^T B_{k,k} \\ B_{pp}^T B_{k,k-1} & B_{pp}^T B_{kk} \end{pmatrix} \quad . \tag{5.10}$$

The corresponding terms in the above matrix are:

$$B_{p-1,p-1}^T B_{k-1,k-1} =$$

$$= \begin{pmatrix} g^{1,(p-1)} & g^{2,(p-1)} & \dots & g^{n,(p-1)} \\ g^{1,p} & g^{2,p} & \dots & g^{n,p} \\ \dots & \dots & \dots & \dots \\ g^{1,n} & g^{2,n} & \dots & g^{nn} \end{pmatrix} \begin{pmatrix} g^{1,(k-1)} & g^{1,k} & \dots & g^{1,n} \\ g^{2,(k-1)} & g^{2,k} & \dots & g^{2,n} \\ \dots & \dots & \dots & \dots \\ g^{n,k-1} & g^{n,k} & \dots & g^{nn} \end{pmatrix} =$$

$$= \begin{pmatrix} P_{11} & P_{12} & \dots & P_{n-k+2} \\ \dots & \dots & \dots & \dots \\ \dots & \dots & P_{sr\dots} & \dots \\ P_{n-p+2,1} & P_{n-p+2,2} & \dots & P_{n-p+2,n-k+2} \end{pmatrix} , \tag{5.11}$$

where all the elements of the matrix are nonzero and the element P_{sr} on the s−th row and on the r−th column is equal to $P_{sr} \equiv \sum_{i=1}^{n} g^{i,p+s-2} g^{i,k+r-2}$,

$$B_{p,p-1}^T B_{k,k-1} = \begin{pmatrix} 0 & 0 & .. & 0 \\ 0 & \sum_{i=1}^{n} g^{(p-1),i} g^{(k-1),i} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 \end{pmatrix} \tag{5.12}$$

where $B_{p,p-1}^T$ and $B_{k,k-1}$ are $(n-p+2) \times n$ and $n \times (n-k+2)$ matrices and the resulting $n \times n$ matrix has only one nonzero element $\sum_{i=1}^{n} g^{(p-1),i} g^{(k-1),i}$ on the second row and on the second column. Next let us calculate the matrix $B_{pp}^T B_{k,k-1}$, which is a product of the $(n-p+1) \times n$ matrix B_{pp}^T and the $n \times (n-k+2)$ matrix $B_{k,k-1}$ with the only nonzero second column:

$$B_{pp}^T B_{k,k-1} = \begin{pmatrix} 0 & F_{12} & \dots & 0 \\ 0 & F_{22} & \dots & 0 \\ \dots & .. & \dots & \dots \\ 0 & F_{n-p+1,2} & \dots & 0 \end{pmatrix} \quad . \tag{5.13}$$

The resulting matrix $B_{pp}^T B_{k,k-1}$ has a dimension $(n-p+1) \times (n-k+2)$ with the only nonzero second column with an element on the $r-$th row and on the $2-$nd column

$$F_{r2} \equiv \sum_{i=1}^{n} g^{i,(p+r-1)} g^{k-1,i}.$$

Next let us find the matrix $B_{p,p-1}^T B_{k,k}$, which is a product of the $(n-p+2) \times n$ matrix $B_{p,p-1}^T$ and the $n \times (n-k+1)$ matrix $B_{k,k}$:

$$B_{p,p-1}^T B_{k,k} =$$

$$= \begin{pmatrix} 0 & .. & 0 & .. & 0 \\ \sum_{i=1}^{n} g^{(p-1),i} g^{i,k} & .. & \sum_{i=1}^{n} g^{(p-1),i} g^{i,(k+s-1)} & .. & \sum_{i=1}^{n} g^{(p-1),i} g^{i,n} \\ .. & ... & .... & ... & .. \\ 0 & .. & 0 & .. & 0 \\ 0 & .. & 0 & .. & 0 \end{pmatrix} , \qquad (5.14)$$

where in the last formulae we have used that the obtained matrix has $(n-k+1)$ columns and therefore the indice $k+s-1$ in the expression for the element $\sum_{i=1}^{n} g^{(p-1),i} g^{i,(k+s-1)}$ on the $2-$nd row and $s-$th column ranges from k to $k+s-1 = k+(n-k+1)-1 = n$.

It remains only to calculate the matrix $B_{pp}^T B_{kk}$, but it is the same as (5.11), this time with an element

$$P_{sr} \equiv \sum_{i=1}^{n} g^{i,p+s-1} g^{i,k+r-1} \qquad (5.15)$$

on the $s-$th row and on the $r-$th column.

Using the above developed techniques for matrix multiplication, let us calculate the $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ matrix $B^T B$ (recall - B is an $n^2 \times \frac{n(n+1)}{2}$ matrix and B^T is an $\frac{n(n+1)}{2} \times n^2$ matrix), which is the $n-$dimensional analogue of the matrix (3.8). Taking into consideration (5.9), one has

$$B^T B = \begin{pmatrix} \sum_{i=1}^{n} B_{i1}^T B_{i1} & \sum_{i=2}^{n} B_{i1}^T B_{i2} & ... & \sum_{i=n}^{n} B_{i1}^T B_{in} \\ \sum_{i=2}^{n} B_{i2}^T B_{i1} & \sum_{i=2}^{n} B_{i2}^T B_{i2} & ... & \sum_{i=n}^{n} B_{i2}^T B_{ni} \\ .... & .. & ... & ... \\ \sum_{i=n}^{n} B_{in}^T B_{i1} & .. & ... & \sum_{i=n}^{n} B_{in}^T B_{in} \end{pmatrix} . \qquad (5.16)$$

The above matrix contains three types of elements:

1-st type. Elements below the block diagonal of the type $\sum_{\alpha=j}^{n} B_{\alpha j}^T B_{\alpha k}$ with $k < j$. Carrying out the matrix multiplication and for the moment not taking the summation

10

over α, we find

$$B_{\alpha j}^T B_{\alpha k} = \begin{pmatrix} 0 & 0 & & .. & & 0 & 0 \\ .. & .. & & .. & & .. & .. \\ 0 & .. & \sum_{i=1}^{n} g^{ji} g^{ki} & .. & & 0 \\ 0 & .. & & .. & & .. & .0 \\ 0 & 0 & & .. & & .. & 0 \end{pmatrix} \quad , \tag{5.17}$$

where the only nonzero element $\sum_{i=1}^{n} g^{ji} g^{ki}$ in the matrix is on the $(\alpha - j + 1)$ row and on the $(\alpha - k + 1)$ column and the matrices $B_{\alpha j}^T$ and $B_{\alpha k}$ are of dimensions $(n - j + 1) \times n$ and $n \times (n - k + 1)$ correspondingly.

Since the first term in the sum $\sum_{\alpha=j}^{n} B_{\alpha j}^T B_{\alpha k}$ will be $B_{\alpha \alpha}^T B_{\alpha k}$, let us find it, performing the same kind of matrix multiplication as in (5.13):

$$B_{\alpha\alpha}^T B_{\alpha k} = \begin{pmatrix} 0 & 0 & \sum_{i=1}^{n} g^{i\alpha} g^{ki} & 0 & 0 \\ .. & .. & & .. & & .. & .. \\ 0 & .. & \sum_{i=1}^{n} g^{i,\alpha+r-1} g^{k,i} & .. & 0 \\ 0 & .. & & .. & & .. & .0 \\ 0 & 0 & \sum_{i=1}^{n} g^{i,n} g^{k,i} & & .. & 0 \end{pmatrix} \quad , \tag{5.18}$$

where the only nonzero column is the $(\alpha - k + 1)$ one and the element in this column and on the $r-$th row is $\sum_{i=1}^{n} g^{i,\alpha+r-1} g^{k,i}$. The matrices $B_{\alpha\alpha}^T$ and $B_{\alpha k}$ are of dimensions $(n - \alpha + 1) \times n$ and $n \times (n - k + 1)$ correspondingly and the resulting matrix $B_{\alpha\alpha}^T B_{\alpha k}$ is $(n - \alpha + 1) \times (n - k + 1)$ dimensional.

2-nd type. Elements above the block diagonal of the type $\sum_{\alpha=k}^{n} B_{\alpha j}^T B_{\alpha k}$ with $k > j$ [The summation indice α takes at first the value of that indice (k or j), which is greater than the other]. The first term in the above sum is $B_{\alpha j}^T B_{\alpha \alpha}$, which can be found to be

$$B_{\alpha j}^T B_{\alpha \alpha} =$$

$$= \begin{pmatrix} 0 & 0 & .. & 0 & 0 \\ .. & .. & .. & .. & .. \\ \sum_{i=1}^{n} g^{j,i} g^{1,(\alpha+i-1)} & ... & \sum_{i=1}^{n} g^{j,i} g^{p,(\alpha+i-1)} & .. & \sum_{i=1}^{n} g^{j,i} g^{n,(\alpha+i-1)} \\ .. & .. & ... & .. & .. \\ 0 & 0 & ... & 0 & 0 \end{pmatrix} . \tag{5.19}$$

The above $(n - j + 1) \times (n - \alpha + 1)$ matrix $B_{\alpha j}^T B_{\alpha \alpha}$ contains a nonzero $(\alpha - j + 1)$ row

with an element in the $p-$th column, equal to

$$K_{\alpha-j+1,p} = \sum_{i=1}^{n} g^{j,i} g^{p,(\alpha+i-1)} \qquad . \tag{5.20}$$

3-rd type. Elements situated on the block diagonal of the type

$$\sum_{\alpha=k}^{n} B_{\alpha k}^{T} B_{\alpha k} = B_{\alpha\alpha}^{T} B_{\alpha\alpha} + \sum_{\alpha=k+1}^{n} B_{\alpha k}^{T} B_{\alpha k} \quad . \tag{5.21}$$

Similarly to the calculation of (5.15), the first term in (5.21) ($B_{\alpha\alpha}^{T}$ is an $(n - \alpha + 1) \times n$ matrix ; $B_{\alpha\alpha}$ is an $n \times (n-\alpha+1)$ matrix) is found to be the following $(n-\alpha+1) \times (n-\alpha+1)$ matrix (also, $\alpha = k$) :

$$B_{\alpha\alpha}^{T} B_{\alpha\alpha} = \begin{pmatrix} N_{11} & N_{12} & ... & N_{1,(n-\alpha+1)} \\ N_{21} & N_{22} & ... & N_{2,(n-\alpha+1)} \\ ... & ... & ... & .. \\ N_{n-\alpha+1,1} & ... & ... & N_{n-\alpha+1,n-\alpha+1} \end{pmatrix} \quad , \tag{5.22}$$

where

$$N_{pq} = \sum_{i=1}^{n} g^{i,\alpha+p-1} g^{i,\alpha+q-1} \quad . \tag{5.23}$$

The second term in (5.21) is in fact the $(n - k + 1) \times (n - k + 1)$ matrix (5.17) for $j = k$. The summation over α from $\alpha = k + 1$ to n will give a diagonal $(n - k + 1) \times (n - k + 1)$ matrix with an element $G_{\alpha-k+1,\alpha-k+1} = \sum_{i=1}^{n} (g^{k,i})^2$ on the $(\alpha - k + 1)$ row and on the $(\alpha - k + 1)$ column, which will be situated on the main (block) diagonal from $\alpha = k + 1$ to n. Since for $\alpha = k + 1$ we have $\alpha - k + 1 = k + 1 - k + 1 = 2$ and for $\alpha = n$ we have also $\alpha - k + 1 = n - k + 1$, this means that the matrix $\sum_{\alpha=k+1}^{n} B_{\alpha k}^{T} B_{\alpha k}$ will have the following structure:

$$\sum_{\alpha=k+1}^{n} B_{\alpha k}^{T} B_{\alpha k} = \begin{pmatrix} 0 & 0 & .. & 0 & 0 \\ 0 & \sum_{i=1}^{n}(g^{k,i})^2 & .. & 0 & 0 \\ 0 & 0 & \sum_{i=1}^{n}(g^{k,i})^2 & .. & 0 \\ ... & ... & .... & .... & ... \\ 0 & 0 & .... & 0 & \sum_{i=1}^{n}(g^{k,i})^2 \end{pmatrix} \quad . \tag{5.24}$$

Therefore, summing up the matrices (5.22) and (5.25), one obtains the general structure

of the matrices (5.21) $\sum_{\alpha=k}^{n} B_{\alpha k}^T B_{\alpha k}$ on the block diagonal:

$$
\sum_{\alpha=k}^{n} B_{\alpha k}^T B_{\alpha k} = \begin{pmatrix} N_{11} & N_{12} & ... & N_{1,(n-\alpha+1)} \\ N_{21} & N_{22} + G_{22} & ... & N_{2,(n-\alpha+1)} \\ ... & ... & ... & .. \\ N_{n-\alpha+1,1} & ... & ... & N_{n-\alpha+1,n-\alpha+1} + G_{22} \end{pmatrix} , \tag{5.25}
$$

where N_{pq} and G_{22} are given by expressions (5.23) and $G_{2,2} = \sum_{i=1}^{n}(g^{k,i})^2$ respectively. Consequently (for $p \geq 2$)

$$
N_{pp} + G_{22} = \sum_{i=1}^{n} \left[g^{i,(k+p-1)} g^{i,(k+p-1)} + (g^{k,i})^2 \right] . \tag{5.26}
$$

Let us find now the structure of the off - diagonal block matrices, situated below the diagonal in the block matrix (5.16). Each such an element can be decomposed as

$$
\sum_{\alpha=j(j>k)} B_{\alpha j}^T B_{\alpha k} = B_{\alpha \alpha}^T B_{\alpha k} + \sum_{\alpha=j+1}^{n} B_{\alpha j}^T B_{\alpha k} . \tag{5.27}
$$

The first term in (5.27) is the already known $(n-\alpha+1) \times (n-\alpha+1)$ matrix (5.18) with the only nonzero $(j-k+1)$ column. The second term is the sum from $\alpha = j+1$ to $\alpha = n$ of the $(n-\alpha+1) \times (n-k+1)$ matrices $B_{\alpha j}^T B_{\alpha k}$, already calculated in (5.17) and having the only nonzero element $\sum_{i=1}^{n} g^{ji} g^{ki}$ on the $(\alpha-j+1)$ row and on the $(\alpha-k+1)$ column. The summation over α from $\alpha = j+1$ to $\alpha = n$ means that in the sum $\sum_{\alpha=j+1}^{n} B_{\alpha j}^T B_{\alpha k}$ the element $\sum_{i=1}^{n} g^{ji} g^{ki}$ will appear beginning from the row $\alpha-j+1 = 2$ (for $\alpha = j+1$) up to the row $\alpha - j + 1 = n - j + 1$ (for $\alpha = n$), which in fact is the last row. Correspondingly, the same element will appear beginning from the column $\alpha - k + 1 = j - k + 2$ (for $\alpha = j+1$) and ending at the column $\alpha - k + 1 = n - k + 1$ (at $\alpha = n$), which is the last column. In other words, the summation of the matrices $B_{\alpha j}^T B_{\alpha k}$, containing one element, effectively results in a matrix, filled up from the second row to the end and from the $(j - k + 2)$ column to the end:

$$
\sum_{\alpha=j+1}^{n} B_{\alpha j}^T B_{\alpha k} = \sum_{\alpha=j+1}^{n} \begin{pmatrix} 0 & .. & 0 & .. & 0 \\ ... & .. & .... & ... & ... \\ 0 & .... & \sum_{i=1}^{n} g^{ji} g^{ki} & .... & 0 \\ ... & ... & ... & ... & ... \\ 0 & .. & 0 & ... & 0 \end{pmatrix} =
$$

13

$$
= \begin{pmatrix}
0 & \ldots & \text{column } (j-k+2)\ldots & 0 & & 0 \\
0 & \ldots & \sum\limits_{i=1}^{n} g^{ji}g^{ki} & & .. & \sum\limits_{i=1}^{n} g^{ji}g^{ki} \\
.. & .. & .. & & .. & .. \\
0 & .. & \sum\limits_{i=1}^{n} g^{ji}g^{ki} & & .. & \sum\limits_{i=1}^{n} g^{ji}g^{ki} \\
0 & .. & \sum\limits_{i=1}^{n} g^{ji}g^{ki} & & .. & \sum\limits_{i=1}^{n} g^{ji}g^{ki}
\end{pmatrix} . \tag{5.28}
$$

Now recall that the matrix $B_{\alpha\alpha}^T B_{\alpha k}$ (5.18) had a nonzero $(j-k+1)$ column, so therefore the structure of the whole matrix $\sum\limits_{\alpha=j}^{n} B_{\alpha j}^T B_{\alpha k}$ $(j>k)$ below the diagonal is similar to (5.28), but with the additional $(j-k+1)$ column:

$$
\begin{pmatrix}
0 & 0.......... & \sum\limits_{i=1}^{n} g^{i,\alpha}g^{k,i} & 0 & .. & 0 \\
0 & 0...... & \sum\limits_{i=1}^{n} g^{i,\alpha+1}g^{k,i} & \sum\limits_{i=1}^{n} g^{ji}g^{ki} & \ldots & \sum\limits_{i=1}^{n} g^{ji}g^{ki} \\
\ldots & \ldots & ..... & \ldots & \ldots & \ldots \\
0 & 0........ & \sum\limits_{i=1}^{n} g^{i,\alpha+r-1}g^{k,i} & \sum\limits_{i=1}^{n} g^{ji}g^{ki} & .. & \sum\limits_{i=1}^{n} g^{ji}g^{ki} \\
.. & ..... & .. & .. & \ldots & .. \\
0 & 0..... & \sum\limits_{i=1}^{n} g^{i,n}g^{k,i} & \sum\limits_{i=1}^{n} g^{ji}g^{ki} & .. & \sum\limits_{i=1}^{n} g^{ji}g^{ki}
\end{pmatrix} . \tag{5.29}
$$

In a completely analogous way the elements above the block diagonal in (5.16) can be found. These elements $\sum\limits_{\alpha=k}^{n} B_{\alpha j}^T B_{\alpha k}$ $(k>j)$ can be decomposed as

$$
\sum\limits_{\alpha=k(k>j)} B_{\alpha j}^T B_{\alpha k} = B_{kj}^T B_{kk} + \sum\limits_{\alpha=k+1}^{n} B_{\alpha j}^T B_{\alpha k} \quad . \tag{5.30}
$$

This formulae is similar to (5.27) for the below - diagonal elements, but here in (5.30) we have the matrix $B_{kj}^T B_{kk}$ instead of the matrix $B_{\alpha\alpha}^T B_{\alpha k}$ and the summation over the indice α in the second term is from $\alpha=k+1$ to $\alpha=n$ instead of $\alpha=j+1$ to $\alpha=n$ in (5.27). The first term in (5.30) $B_{kj}^T B_{kk}$ is the $(n-j+1)\times(n-k+1)$ matrix (5.19) (with $\alpha=k$) with the only nonzero $(k-j+1)$ row with the elements

$$
\sum\limits_{i=1}^{n} g^{ji}g^{1,(k+i-1)} \quad ; \quad \ldots \sum\limits_{i=1}^{n} g^{ji}g^{p,(k+i-1)}; \ldots \sum\limits_{i=1}^{n} g^{ji}g^{n,(k+i-1)} \tag{5.31}
$$

The second term in (5.30) is again the sum of the matrices (5.17). The only nonzero element $\sum\limits_{i=1}^{n} g^{ji}g^{ki}$ will now appear in the final sum from the $\alpha-j+1=k-j+2$ row $(\alpha=k+1)$ until the $k-j+1=n-j+1$ $(\alpha=n)$ row, which is the last one. Also,

the same element will fill up the columns from $\alpha - k + 1 = 2$ $(\alpha = k + 1)$ to the column $\alpha - k + 1 = n - k + 1$ $(\alpha = n)$, which is also the last one. Therefore, summing up the two terms in (5.30), one obtains the following matrix for the above - diagonal block terms, in which the $(k - j + 1)$ row is filled up with the elements $\sum_{i=1}^{n} g^{j,i} g^{p,(\alpha+i-1)}$ and from the next $(k - j + 2)$ row to the end and from the second column to the end column the matrix is filled up with the other element $\sum_{i=1}^{n} g^{ji} g^{ki}$:

$$
\begin{pmatrix}
0 & 0......... & 0.. & 0.. & .. & 0 \\
.. & ...... & ... & ... & ... & .... \\
\sum_{i=1}^{n} g^{j,i} g^{1,(\alpha+i-1)} & \sum_{i=1}^{n} g^{j,i} g^{2,(\alpha+i-1)} & .... \sum_{i=1}^{n} g^{j,i} g^{p,(\alpha+i-1)} & ... & ... & \sum_{i=1}^{n} g^{j,i} g^{n,(\alpha+i-1)} \\
0 & \sum_{i=1}^{n} g^{ji} g^{ki} & ..... & \sum_{i=1}^{n} g^{ji} g^{ki} & .. & \sum_{i=1}^{n} g^{ji} g^{ki} \\
.. & ..... & .. & .. & ... & .. \\
0 & \sum_{i=1}^{n} g^{ji} g^{ki} & \sum_{i=1}^{n} g^{i,n} g^{k,i} & \sum_{i=1}^{n} g^{ji} g^{ki} & .. & \sum_{i=1}^{n} g^{ji} g^{ki}
\end{pmatrix}
$$

$$(5.32)$$

Let us now summarize the obtained results for the $n-$dimensional case. The (predetermined) system of equations $g_{ij} g^{jk} = \delta_i^k$ was represented as $BY = \overline{1}$, where $\overline{1}$ is an n^2 dimensional vector, whose transponed one is defined as $\overline{1}^T = (\overline{1}^{T1}, \overline{1}^{T2}, ..., \overline{1}^{Tn})$ and the corresponding transponed $n-$ dimensional vectors $\overline{1}^{T1}, \overline{1}^{T2}, ..., \overline{1}^{Tn}$ are defined as follows: $\overline{1}^{T1} = (1, 0, ..., 0), \overline{1}^{T2} = (0, 1, 0, ...0)$ and the $k-$th transponed vector $\overline{1}^{Tk} = (0, 0, ..., 1, 0, ...0)$ contains the number 1 on the $k-$th place. The $n^2 \times \frac{n(n+1)}{2}$ matrix B in terms of the elementary block matrices B_{ij} and the $\frac{n(n+1)}{2}$ dimensional vector Y were defined by formulaes (5.5 - 5.8) and (5.4) respectively. In order to solve the system, we multiplied it to the left with the transponed matrix B^T and thus the solution for the vector Y in matrix notations can be found as $Y = (B^T B)^{-1} B^T \overline{1}$, where the expression for $B^T \overline{1}$ can easily be found to be

$$
B^T \overline{1} = \begin{pmatrix}
B_{11}^T & B_{21}^T & ...... & B_{n1}^T \\
0 & B_{22}^T & .... & B_{n2}^T \\
... & ... & ... & .. \\
0 & 0 & .. & B_{nn}^T
\end{pmatrix}
\begin{pmatrix}
\overline{1}^1 \\
\overline{1}^2 \\
... \\
\overline{1}^m
\end{pmatrix}
= \begin{pmatrix}
\sum_{i=1}^{n} B_{i1}^T \overline{1}^i \\
\sum_{i=1}^{n} B_{i2}^T \overline{1}^i \\
... \\
\sum_{i=1}^{n} B_{im}^T \overline{1}^i
\end{pmatrix}
\qquad . \qquad (5.33)
$$

The expressions for the elements of the vector in the R. H. S. can also be found, but this will not be performed here and will be left for the interested reader. Let us remind again that each "element" of this vector is in itself an $n-$dimensional vector and $m = \frac{n(n+1)}{2}$ gives the number of the "block" elements of this vector - column. The number m should

be an integer number, but this requirement can be fulfilled and this will be commented in the next subsection.

The main result of this subsection are contained in the expressions (5.25), (5.29) and (5.32) for the elements of the matrix $B^T B$ on the block diagonal, below the block diagonal and above the block diagonal respectively. In such a way the detailed structure of the matrtix $B^T B$ is known in terms of the elementary constituent block matrices B_{ij}. The structure of the matrix $B^T B$ is important for the following reasons:

1. If one chooses the contravariant metric tensor in the form of a factorized product $\widetilde{g}^{ij} = dX^i dX^j$, then one can answer the question whether it is possible the rows of the matrix $B^T B$ to have some common multiplier. But from the above expressions and since in each "cell" of the elementary block matrices one has a summation of the kind $\sum\limits_{i=1}^{n} g^{i,n} g^{k,i}$, it is evident that such a common multiplier does not exist and therefore the rank of the matrix $B^T B$ cannot be lowered due to the above made choice of g^{ij}.

2. The quadratic structure of the matrix $B^T B$ gives an opportunity to apply the Frobenius formulae for finding the inverse matrix $(B^T B)^{-1}$, as previously discussed.

6 BLOCK MATRIX REPRESENTATION OF THE HOMOGENEOUS SYSTEM OF EQUATIONS WITH A ZERO R. H. S.

Earlier it was shown that for the case $n = 3$ the sub - system of equations $g_{ij} g^{jk} = 0$ with a zero R. H. S. has a determinant of coefficient (functions), equal to zero. The question which naturally arises is whether this property is valid only for the $n = 3$ case and is it valid for the $n-$ dimensional case?

Below it shall be proved that this can be done for the general case. Namely, it shall be established that from the $n^2 - n$ equations with a zero R. H. S. there may be chosen a sub-system of $\binom{n}{2} + n = \frac{n(n+1)}{2}$ equations with a determinant of coefficients, equal to zero. It shall be stressed that the proof will be that such a system exists (i.e. can be chosen) and not that all other choices of the subsystem of equations will also satisfy this requirement. In fact, investigating under what other choices of the sub - system of equations with a zero R. H. S. this property will be preserved, represents an interesting problem for further research.

For the purpose, let us again use the block matrix representation (5.5). Then, in order to obtain the system of equations with a zero R. H. S. , one has to remove the first row (for $i = k = 1$) from the system $g^{nk} g_{ki} = \delta_i^n$ of the first n equations (i.e. for $n = i$), then the second row from the second system of n equations and so on, one has to remove the $k-$th row from the $k-$th system of n equations. Further, in order to receive again a block matrix structure with (elementary) submatrices with n rows, one should add the first row of the second system of n equations as the last row of the first system of equations. In

effect, since the first two rows of the second system have been removed, one should add the first two rows of the third system of n equations as the last two rows of the second system. Therefore, since also the third row in the third system (for $i = k = 3$) has been removed, it has a total of three rows removed, and subsequently three last rows have to be added from the fourth system. Continuing in the same manner, from the $k-$th matrix B_{kk} on the block diagonal we have the first $k-$ rows from B_{kk} removed and also k - last rows added, which should be taken from the below - diagonal matrix $B_{k+1,k}$. Since according to (5.8) the $n \times (n-k+1)$ matrix B_{sk} has a nonzero $(s-k+1)$ column, the matrix $B_{k+1,k}$ will have a nonzero $k+1-k+1 = 2$ column. Therefore the new transformed in this way $n \times (n-k+1)$ matrix, which will be denoted as $\widetilde{B}_{kk}$, will have the following structure:

$$\widetilde{B}_{kk} \equiv \begin{pmatrix} g^{k+1,k} & g^{k+1,k+1} & .. & .. & ... & .. & g^{k+1,n} \\ g^{k+2,k} & g^{k+2,k+1} & .. & .. & . & . & g^{k+2,n} \\ .. & .. & .. & .. & . & .. & ... \\ g^{nk} & g^{n,(k+1)} & .. & .. & .. & .. & g^{nn} \\ 0 & g^{k1} & .. & 0 & .. & 0 & 0 \\ .. & & .. & .. & .. & .. & .. \\ 0 & g^{kk} & .. & 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.1}$$

In the same way, the below - diagonal transformed matrix $\widetilde{B}_{k+1,k}$, obtained from $B_{k+1,k}$ by removing its first k and adding k rows from $B_{k+2,k}$, will be of the following kind:

$$\widetilde{B}_{k+1,k} \equiv \begin{pmatrix} 0 & g^{k,(k+1)} & 0 & .. & 0 & 0 \\ .. & .. & .. & .. & .. & .. \\ 0 & g^{k,n} & 0 & .. & 0 & 0 \\ 0 & 0 & g^{k,1} & .. & 0 & 0 \\ .. & .. & .. & ... & .. & .. \\ 0 & 0 & g^{k,k} & .. & 0 & 0 \end{pmatrix} . \tag{1}$$

Since the block matrix (5.5) has a **t**riangular structure, for our further purposes only the structure of the block - diagonal matrices $\widetilde{B}_{kk}$ will be relevent.

Next, our goal will be to divide the the block - matrix (5.5) into elemetary block matrices with an equal number (n) of rows and columns. Let us remind once again that the block - matrix (5.5) contained elementary block - matrices with an unequal number of columns - $n, (n-1), (n-2)$......correspondingly. For the purpose, we shall take one (left) column from the block matrix with $(n-2)$ columns and transfer it to the left to the block matrice with $(n-1)$ columns. As a result, the block matrices on the second block matrix column (B. M. C.) will already contain n columns. Since in the block matrix column one column has been transfered, one has to add 3 columns from the $(n-3)-$rd block matrix column to the $(n-2)-$nd block matrix column in order to obtain again a block matrix column, consisting of elementary $n \times n$ matrices. Continuing in the same manner with the $(n-3)-$rd B. M. C., one has to add to its right end 6 columns from the $(n-4)-$th B. M. C. Now let us write down the numbers of the corresponding block columns and below with a $(-)$ sign the number of columns, transferred to the (neighbouring) B. M. C.

; with a (+) sign the number of columns, joined to the B. M.C. (to the right side) from the neighbouring (right) B. M. C.

$(n-1)$; $(n-2)$; $(n-3)$; $(n-4)$; $(n-5)$

$+1$; -1 $+3$; -3 $+6$; -6 $+10$; -10 $+15$.

Now let us look at the numbers with a minus sign, which form the following number sequence (with the corresponding number in the sequence denoted):

$$1, 3, 6, 10, 15, 21......$$
$$1, 2, 3, 4, 5, 6, ... \tag{6.2}$$

It is trivial to note that each number in the sequence (upper row) is in fact a sum of the corresponding numbers in the lower row up to that number. For example, the number 21 in the sequence (upper row) can be represented as a sum of the numbers in the sequence (lower row): $21 = 1 + 2 + 3 + 4 + 5 + 6$. The same with the number 10 $\implies 10 = 1+2+3+4$. Therefore, to the sequence number k in the low row will correspond

the number $\frac{k(k+1)}{2}$ in the upper row, which is the sum of the first k numbers in the low row. Since the number k in the lower (6.2) corresponds to the $(n-k)-$th block column, the corresponding number will be $\frac{k(k-1)}{2}$, and it will correspond to the number of columns, which have to be removed from the diagonal block matrix $B_{k+1,(k+1)}$. At the same time, to the right end one should add $k + \frac{k(k-1)}{2} = \frac{k(k+1)}{2}$ columns. This number is exactly equal to the number of left columns, removed from the (right) neighbouring matrix $B_{(k+1),(k+2)}$. This can serve also as a consistency check that the performed calculation is consistent and correct.

From the matrix (6.1) for $\widetilde{B}_{k+1,(k+1)}$ we have to delete the first $\overline{s}+1$ left columns, the first (upper) elements of which begin with the elements $g^{(k+2),(k+1)}; g^{(k+2),(k+2)}.............,g^{(k+2),(\overline{s}+k)}$, where $\overline{s} = \frac{k(k-1)}{2}$. Now let us denote by $\overline{B}_{(k+1),(k+1)}$ the matrix $\widetilde{B}_{k+1,(k+1)}$ with $\frac{k(k+1)}{2}$ left columns removed and $\frac{k(k+1)}{2}$ right columns added. The upper elements of the (left) remaining columns will be $g^{(k+2),(\overline{s}+k+1)}, g^{(k+2),(\overline{s}+k+2)}, ...,g^{(k+2),n}$, where

$$\overline{s} + k + 1 = \frac{k(k-1)}{2} + k + 1 = \frac{k(k+1)}{2} + 1 \tag{6.3}$$

and the matrix $\overline{B}_{(k+1),(k+1)}$ will contain $n - \frac{k(k+1)}{2}$ left nonzero columns. Therefore the $n \times n$ matrix $\overline{B}_{(k+1),(k+1)}$ will have the following structure:

$$\overline{B}_{(k+1),(k+1)} \equiv \begin{pmatrix} L_1 & L_2 \\ L_3 & L_4 \end{pmatrix} \quad , \tag{6.4}$$

where the $(n - k + 1) \times \left[n - \frac{k(k+1)}{2} \right]$ matrix L_1 is

$$L_1 \equiv \begin{pmatrix} g^{(k+2),(\frac{k(k+1)}{2}+1)} & ... & ... & g^{(k+2),n} \\ g^{(k+3),(\frac{k(k+1)}{2}+1)} & ... & .... & g^{(k+3),n} \\ .... & ... & ... & ... \\ g^{n,\frac{k(k+1)}{2}+1} & ... & ... & g^{n,n} \end{pmatrix} \tag{6.5}$$

18

and L_2, L_3 and L_4 are zero matrices of dimensions $(n - k + 1) \times \left[\frac{k(k-1)}{2}\right], (k + 1) \times \left[n - \frac{k(k+1)}{2}\right]$ and $(k + 1) \times \left[\frac{k(k+1)}{2}\right]$ correspondingly. Note also that the block matrices L_1 and L_3 contain $\left[n - \frac{k(k+1)}{2}\right]$ nonzero columns, and the remaining $\left[\frac{k(k+1)}{2}\right]$ columns of the matrices L_2 and L_4 are exactly equal to the number of zero columns, added to the right side of the matrix $\overline{B}_{(k+1),(k+1)}$ from the neighbouring matrix $\overline{B}_{(k+1),(k+2)}$. Since this result depends on the initial structure of the matrix $B_{(k+1),(k+1)}$ and on the expression (6.3) (which are both independent on the number of removed right columns), this also confirms the consistency of the calculation.

Note that the block structure of the matrix (6.4) has been revealed on the base of the assumption that the elements $g^{(k+1),1}, ...g^{(k+1),(k+1)}$ in the last $(k+1)$ rows and the second column in the matrix (6.1) will be among the first removed to the left (and outside the matrix) columns. However, for $\overline{s} = 0$ (i.e. $k = 1$) the elements in the second column of the last two rows of the matrix $\overline{B}_{2,2}$ will contain the elements g^{21}, g^{22}. Therefore, after removing the first column in the matrix (6.1) $\widetilde{B}_{(k+1),(k+1)}$ (for $k = 1$) and adding to the right one zero column, the obtained structure of the $n \times n$ matrix $\overline{B}_{2,2}$ will be the following:

$$
\overline{B}_{2,2} = \begin{pmatrix}
g^{33} & g^{34} & ... & g^{3n} & 0 & 0 \\
g^{43} & g^{44} & ... & g^{4n} & 0 & 0 \\
.. & ... & .. & ... & .. & .. \\
g^{n3} & g^{n4} & ... & g^{nn} & 0 & 0 \\
g^{21} & 0 & .. & 0 & 0 & 0 \\
g^{22} & 0 & ... & 0 & 0 & 0
\end{pmatrix} . \tag{6.6}
$$

Having established the block structure of the matrix of coefficients of the $n-$dimensional predetermined system $g_{ij}g^{jk} = 0$ with a homogeneous zero R. H. S., it is now easy to show that $\frac{n(n+1)}{2}$ equations can be chosen so that the determinant of coefficients will be zero. Let us take for example the first $\frac{n(n+1)}{2}$ equations from the system of $(n^2 - n)$ equations with a zero R. H. S. with the corresponding $\frac{n(n+1)}{2} \times \frac{n(n+1)}{2}$ dimensional block matrix. In the particular case the dimension of the block matrix is determined by the number of block matrices on the horizontal $(\frac{n(n+1)}{2})$ and on the vertical $(\frac{n(n+1)}{2})$. Therefore, outside this matrix will remain a matrix of $\frac{n(n+1)}{2}$ matrix block columns and $(n - 1) - \frac{n(n+1)}{2} = \frac{(n-3)}{2}$ block rows. The last assumption presumes that the spacetime dimension number n is an odd one, so that $(n + 1)$ and $(n - 3)$ are dividable by two. Otherwise, if n is an even number, one may consider $\frac{n}{2} \times \frac{n}{2}$ dimensional elementary block matrices $\overline{B}_{k,k}$. Then the full block matrix of the system will have $(n + 1)$ block matrices on the block horizontal (i.e. $(n + 1)$ block columns) and $2n^2$ matrices on the block vertical (i. e. $2n^2$ block rows). The block matrix of the homogeneous system of equations (with a zero R. H. S.) will be $2(n - 1) \times (n + 1)$ "block" dimensional. The chosen block matrix will be $(n + 1) \times (n + 1)$ "block" dimensional. Outside this matrix there will remain a block matrix of $2n - 2 - n - 1 = n - 3$ block rows and $(n + 1)$ block columns.

Let us now compute the determinant of the triangular matrix (5.5), from which we take the first $\frac{(n+1)}{2}$ (or $\frac{n}{2}$) block rows. This $\frac{(n+1)}{2} \times \frac{(n+1)}{2}$ (or $\frac{n}{2} \times \frac{n}{2}$) block determinant $\widetilde{S}$

will be equal to

$$\widetilde{S} = \prod_{i=1}^{\frac{n+1}{2}} (det\overline{B}_{ii}) \quad . \tag{6.7}$$

But for $i \neq 1, 2$ the expression for $det\overline{B}_{ii}$ has to be found from formulae (6.3) for the block matrix $\overline{B}_{kk}$. Since only one of the submatrices L_1 is different from zero, it is clear that $det\overline{B}_{ii} = detL_1.0 = 0$, therefore the whole expression (6.7) equals zero.

Thus we have proved that by taking $\frac{(n+1)}{2}$ (or $\frac{n}{2}$) consequent block rows from the initial (quadratic and triangular) block matrix, the obtained block matrix will have a zero determinant.

7 DISCUSSION

The transformation of a matrix system of algebraic equations (with the unknown matrix A assumed to be a symmetric one) into a system of linear equations is related to some new problems of algebraic nature. The fact that the initial matrix system is a predetermined one and solutions are not unique should be reflected in the finally obtained matrix $B^T B$, but presently it is not known how the properties of the initial matrix system are "encoded" in this matrix.

It is interesting to mention about some possible physical applications of the proposed mathematical algorithm. In theories with extra dimensions, the graviton's field components $h_{MN}(x)$ in the Kaluza-Klein's expansion under toroidal compactification along the extra-dimensional coordinate y are contained the following fields [3]:

$$\text{the radion} \quad H^k = \frac{1}{k} h_j^{\vec{k}j} \quad , \tag{7.1}$$

$$\text{the scalars} \quad S_{ij}^{\vec{k}} = h_{ij}^{\vec{k}} - \frac{k}{(n-1)}(\eta_{ij} + \frac{k_i k_j}{k^2})H^k \quad , \tag{7.2}$$

$$\text{the vectors} \quad V_{\mu j}^{\vec{k}} = \frac{i}{\sqrt{2}} h_{\mu j}^{\vec{k}} \quad , \tag{7.3}$$

$$\text{the gravitons} \quad G_{\mu\nu}^{\vec{k}} = h_{\mu\nu}^k + \frac{k}{3}(\eta_{\mu\nu} + \frac{\partial_\mu \partial_\nu}{k^2})H^{\vec{k}} \quad . \tag{7.4}$$

These fields constitute the $(4+n) \times (4+n)$ dimensional matrix of the bulk graviton

$$\begin{pmatrix} G_{\mu\nu}^{\vec{k}} & V_{\mu j}^{\vec{k}} \\ V_{\mu j}^{\vec{k}} & S_{ij}^{\vec{k}} \end{pmatrix} \quad , \tag{7.5}$$

where $\vec{k}$ is a $n-$component vector, denoting the Kaluza-Klein (KK) numbers along the various extra dimensions. Clearly the matrix (7.5) is a block-structured one and the elements in it satisfy the following constraints

$$(\Box + \widehat{k}^2)G_{\mu\nu}^{\vec{k}} = 0 \quad ; \quad \partial^\mu G_{\mu\nu}^{\vec{k}} = 0 \quad ; \quad G_\mu^{\mu\vec{k}} = 0 \tag{7.6}$$

20

$$\widehat{k}^j V_{\mu j}^{\overrightarrow{k}} = 0 \quad , \partial^\mu V_{\mu j}^{\overrightarrow{k}} = 0 \ , \tag{7.7}$$

$$k^j S_{ik}^{\overrightarrow{k}} = 0 \ , \ S_k^{\overrightarrow{k} j} = 0 \ . \tag{7.8}$$

The operators $(\square + \widehat{k}^2)$ and ∂^μ may be assumed to have some $k-$representation. Then the system (7.6-7.8) is similar to the homogeneous part (when $i \neq k$) of the investigated system $g_{ij} g^{jk} = \delta_i^k$. Since the constraints are acting on a part of the elements of the block-matrix (7.5), it will be predetermined also.

Making use of the defining equalities (7.1-7.4) for the radion, the scalars, the vectors and the gravitons, the system of equations (7.6-7.8) can be written with respect to the graviton's field components $h_{MN}(x)$. From a physical point of view, a solution in terms of these components may be much more valuable.

The method, however, will require a modification of the approach in the preceeding sections, since there it was assumed that the matrix of the coefficient functions A is a symmetrical one $(g^{ij} = g^{ji})$, while finding out the graviton's components will be based on an initial matrix system, which will be a non-symmetric one.

Acknowledgments

References

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